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Eight-Parameter Renormalization Group for Penrose Lattices

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The Ising model and the bond percolation model are set up with eight parameters on two-dimensional Penrose lattices. The behavior of their phase transition is studied by the use of a real-space renormalization group method. The resulting critical indices suggest that they belong to the universality class of two-dimensional periodic lattices.

KEY WORDS: Penrose lattice; quasicrystal; Ising model; percolation; renormalization group; phase transition.

1. INTRODUCTION

Quasicrystals are attracting wide interest because of their unique structure: There is no periodic (but quasiperiodic) translational order and yet they are not amorphous systems.[1] One of the important questions to be answered is whether quasicrystals belong to the same universality class of periodic lattices.

In previous papers, we reported a real-space renormalization group analysis of the Ising model[2] and the bond percolation model[3] on Penrose lattices. The renormalization scheme we developed was based on the decimation transformation called "half-inflation," in which the kite-dart (KD) lattice is transformed into the rhombus (R) lattice and vice versa. We included four types of interaction along the edges and some of the diagonals of tiles. Within this four-parameter space, the microstructure around a lattice site to be decimated was taken into account statistically.

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Our results for the critical point are in good agreement with computer simulation[4,5] and the critical exponent (thermal for the Ising model and the correlation length for the percolation problem) is not far from the universal value for two dimensions. This suggests that models on the Penrose lattice generally belong to the universal class of periodic lattices. This conjecture is further supported by the recent proof by Choy[6] that a certain Baxter-type model on the Penrose lattice belongs to the universality class of two-dimensional lattices.

The Ising model on the *dual* Penrose lattice has been studied by Godreche and Orland.[7] They introduced eight coupling constants on the dual lattice and wrote down a renormalization group. In this paper we introduce eight types of interactions to the renormalization group treatment as Godreche and Orland did so that the statistical average over the microstructures can be avoided. We determine the critical surface in this eight-dimensional space and obtain the critical exponent. In the next section, we briefly review the method employed in Refs. 2 and 3. Section 3 gives the derivation of the new renormalization group equations and the result of the numerical analysis. The last section contains the summary and discussions.

2. FOUR-PARAMETER RENORMALIZATION GROUP

In Refs. 2 and 3 we assigned four coupling constants $k_1,..., k_4$ to the KD-lattice and another four $r_1,..., r_4$ to the R lattice,

$$\begin{cases} k_1: \text{short edge} \\ k_2: \text{long edge} \\ k_3: \text{short "diagonal" of dart} \\ k_4: \text{short diagonal of kite} \\ r_1: \text{single-arrowed edge} \\ r_2: \text{double-arrowed edge} \\ r_3: \text{short diagonal of thin rhombus} \\ r_4: \text{long diagonal of thick rhombus} \end{cases}$$

For the Ising model, k_i and r_i denote $\exp(-2J_i/kT)$, where J_i is the exchange interaction assigned to the bond, T is the temperature, and k is the Boltzmann constant. For the bond percolation, k_i and r_i denote the bond occupation probabilities.

The self-similarity transformation for the quasilattice is "inflation," which transforms a Penrose lattice to a same *type* of Penrose lattice (but not necessarily exactly the same lattice). The new lattice is spanned by tiles τ times larger than before in linear scale [$\tau \equiv (\sqrt{5}+1)/2$ is the golden

ratio]. We denote the inflation by $\mathcal{I}_R^R$ (for R lattice) and $\mathcal{I}_K^K$ (for KD lattice). An inflation consists of two successive "half-inflations" $\mathcal{I}_K^R$ and $\mathcal{I}_R^K$, which transform KD lattice to R lattice and R lattice to KD lattice, respectively,

$$\mathcal{I}_R^R = \mathcal{I}_K^R \mathcal{I}_R^K, \qquad \mathcal{I}_K^K = \mathcal{I}_R^K \mathcal{I}_K^R \tag{2.1}$$

The renormalization group equations were given for each of the operations $\mathcal{I}_K^R$ and $\mathcal{I}_R^K$ and then combined according to (2.1). In each half-inflation, a lattice site to be decimated is contained in a subunit illustrated in Fig. 1a. Let us restrict our discussion to the Ising model for the moment. The first step is to give a scheme to approximate the partition function for Fig. 1a by that for Fig. 1b. The resulting "effective" coupling constants $e(a, b, c)$, $f(a, b, c)$, and $g(a, b, c)$ were given explicitly in Ref. 2. [The coupling constants e and f are defined to be squares of the usual coupling constants, e.g., $e \equiv \exp(-4J/kT)$.] The next step is to combine the subunits of Fig. 1b together to produce the inflated lattice. Note that the way they are combined in the inflated lattice is not unique and depends on the configuration of the neighborhood. For example, the link $r1$ can be shared by (1) two fat rhombi, (2) a fat rhombus and a thin rhombus, or (3) two thin rhombi. In case 1, $r1$ has contributions of e from both of the fat rhombi and thus $r_1 = k_4 e$. On the contrary, for the cases 2 and 3, the power of e is 1/2 and 0, respectively. The same ambiguity exists for other edges, $r2$, $k1$, and $k2$. In Ref. 2, we calculated the probability of occurrence $P^{(1)-(3)}$ for each case and took the statistical average, e.g., $r_1 = k_4 \exp(P^{(1)} + \frac{1}{2}P^{(2)})$. (It turned out

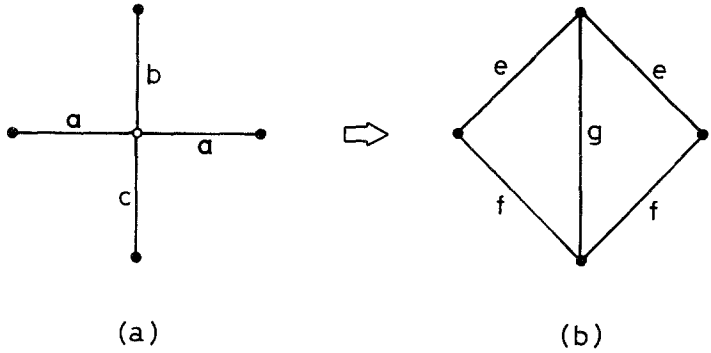

Fig. 1. (a) The subunit before the decimation, which consists of five sites and four bonds. The open circle at the center is to be decimated. The coupling constants a, b, and c denote r's and k's (b) The subunit after the decimation, which consists of four sites and five bonds.

that $P^{(1)} + \frac{1}{2}P^{(2)} = 1/\tau$ for all four edges.) Thus we obtained the following renormalization group equations:

$$\mathcal{J}_{K}^{R} \begin{cases} r_1' = k_4[e(k_1, k_2, k_3)]^{1/\tau} \\ r_2' = k_2[f(k_1, k_2, k_3)]^{1/\tau} \\ r_3' = k_1 \\ r_4' = g(k_1, k_2, k_3) \end{cases}$$

$$\mathcal{J}_{R}^{K} \begin{cases} k_1' = r_2[f(r_1, r_2, r_3)]^{1/\tau} \\ k_2' = r_4[e(r_1, r_2, r_3)]^{1/\tau} \\ k_3' = r_1 \\ k_4' = g(r_1, r_2, r_3) \end{cases} \tag{2.2}$$

For the bond percolation problem,[3] we constructed the transformation from a, b, c to e, f, g, considering the connection probabilities between undecimated sites within the subunits of Fig. 1a and 1b. The renormalized occupation probabilities satisfy equations similar to Eq. (2.2). (See Ref. 3 for details.)

3. EIGHT-PARAMETER RENORMALIZATION GROUP

The statistical average used in the previous method can be avoided by introducing different parameters for different surrounding environments. For the R lattice, we replace the original parameter r_1 assigned to single-arrowed edges by three parameters r_{11}, r_{12}, and r_{13} for the cases 1–3, respectively. We also replace the parameter r_2 assigned to double-arrowed edges by r_{21}, r_{22}, and r_{23} in a similar manner. For the KD lattice, the possible cases for edges $k1$ and $k2$ are links shared by (1) two kites, (2) a kite and a dart, and (3) two darts. Accordingly, k_1 and k_2 are replaced by $k_{11} \sim k_{13}$ and $k_{21} \sim k_{23}$, respectively. Thus, we have eight parameters for each type of the Penrose lattice. The half-inflations are illustrated in Figs. 2 and 3.

3.1. Ising Model

A careful inspection of the inflation rules leads to the following improved renormalization group equations for the Ising model:

$$\mathcal{I}_{\mathrm{K}}^{\mathrm{R}}\begin{cases} r'_{11}=k_4 e_k \\ r'_{12}=k_4 e_k^{1/2} \\ r'_{13}=k_4 \\ r'_{21}=k_{23} f_k \\ r'_{22}=k_{22} f_k^{1/2} \\ r'_{23}=k_{21} \\ r'_3=k_{11} \\ r'_4=g_k \end{cases} \qquad \mathcal{I}_{\mathrm{R}}^{\mathrm{K}}\begin{cases} k'_{11}=r_{23} f_r \\ k'_{12}=r_{22} f_r^{1/2} \\ k'_{13}=r_{21} \\ k'_{21}=r_4 e_r \\ k'_{22}=r_4 e_r^{1/2} \\ k'_{23}=r_4 \\ k'_3=r_{11} \\ k'_4=g_r \end{cases} \tag{3.1}$$

where $\{e,f,g\}_k \equiv \{e,f,g\}(k_{12}, k_{21}, k_3)$ and $\{e,f,g\}_r \equiv \{e,f,g\}(r_{12}, r_{21}, r_3)$. It should be noted that the coupling constants r_{13} and k_{13} are actually irrelevant, since neither of the corresponding configurations is allowed: two thin rhombi cannot share a single-arrowed edge because of the inconsistency of the resulting vertex (this can be checked in the listing of the legal vertices; e.g., Fig. 8 of Ref. 2) and two darts cannot share a shorter edge due to the matching rule (the sharper angles have to meet at the same end). In fact, in Eq. (3.1), the r' do not depend on k_{13} and the k' do not

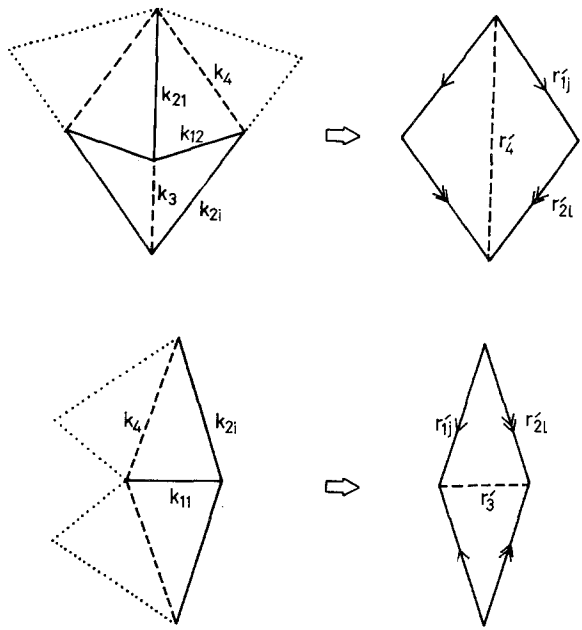

Fig. 2. The half-inflation $\mathcal{I}_{\mathrm{K}}^{\mathrm{R}}$. The indices i, j, and l are determined by the structure around these tiles.

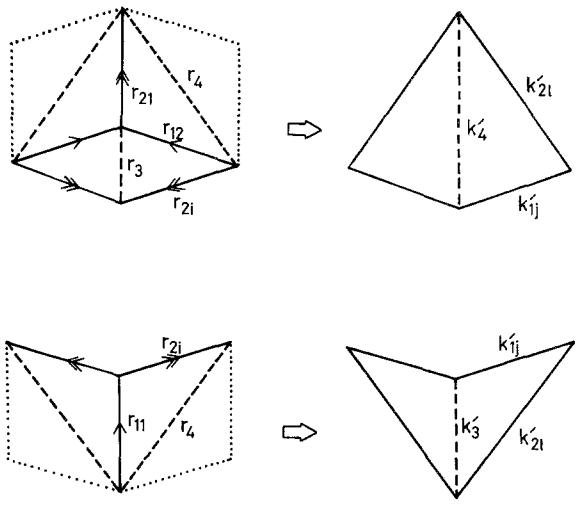

Fig. 3. The half-inflation $\mathscr{I}_{\mathrm{R}}^{\mathrm{K}}$.

depend on r_{13}. The equations for r'_{13} and k'_{13} are purely formal and can be dropped in actual calculation.

We calculated the critical surface in the parameter space. Three-dimensional views of the critical surface in the subspace (r_1, r_2, r_3, r_4), where $r_1 \equiv r_{11} = r_{12}$ and $r_2 \equiv r_{21} = r_{22} = r_{23}$ are shown in Fig. 4. We found that they are almost indistinguishable from the one obtained from Eq. (2.2) (within the thickness of the curves in Ref. 2). This coincidence is quite remarkable if one considers the difference between the renormalization group equations (2.2) and (3.1). For example, the golden ratio τ does not explicitly appear in Eq. (3.1); instead, it is hidden in its structure.

The nontrivial fixed points are at

$$(r) = (0.6396, 0.6908, 0.7461, 0.6115, 0.6246, 0.6379, 0.5504, 0.7124)$$

$$(k) = (0.5504, 0.5801, 0.6115, 0.6379, 0.6741, 0.7124, 0.6396, 0.7461)$$

for R and KD lattices, respectively. The standard procedure for the critical exponent[2,8] yields $y_T = 0.9486$, which leads to the specific heat critical exponent $\alpha = -0.1083$.

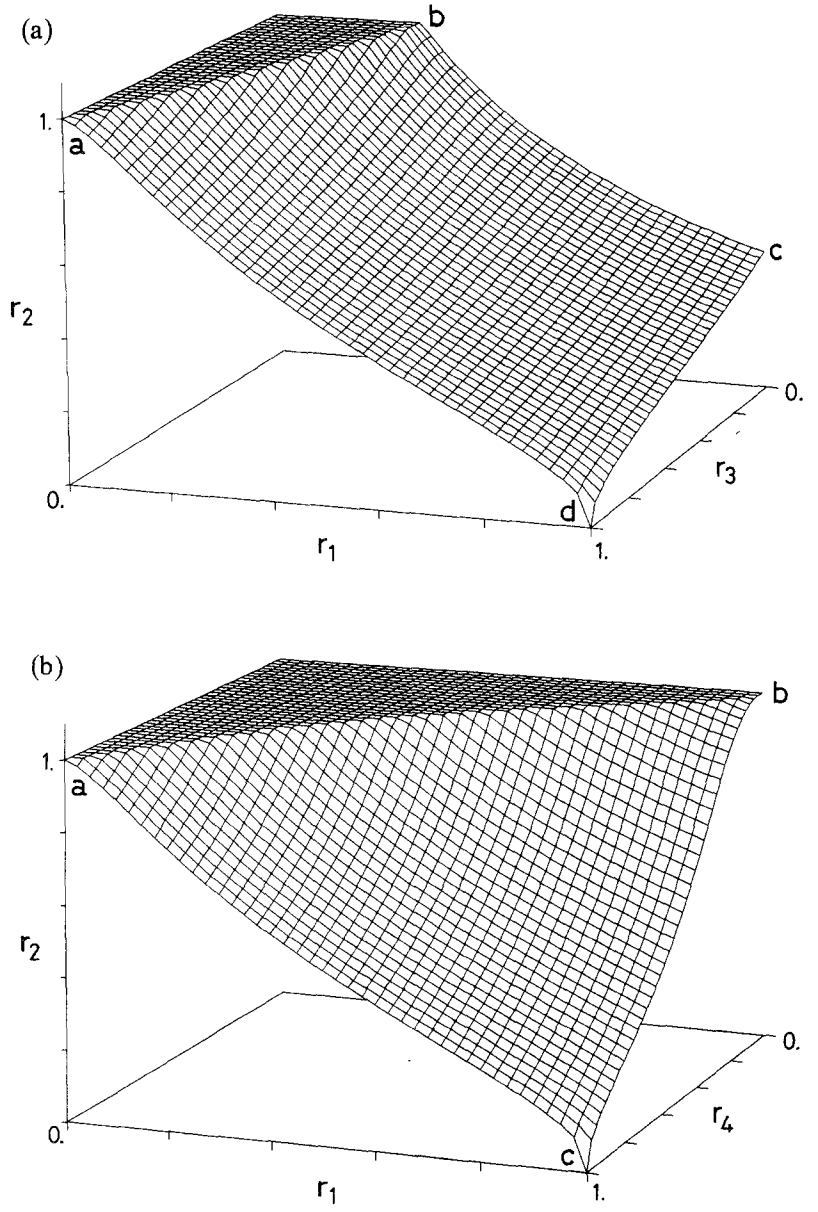

Fig. 4. A perspective view of the critical surface for the Ising model on the R lattice (a) in the (r_1, r_2, r_3) subspace for $r_4 = 1$ and (b) in the (r_1, r_2, r_4) subspace for $r_3 = 1$. ($r_1 \equiv r_{11} = r_{12}$, $r_2 \equiv r_{21} = r_{22} = r_{23}$.) The critical surface consists of the regions *abcd* and *abc* of the shaded area. The remainder of the shaded area are shown only to guide the eye.

3.2. Bond-Percolation Model

For the bond percolation problem, the renormalization group equations read

$$\mathscr{J}_{\mathrm{K}}^{\mathrm{R}}\begin{cases} r'_{11}=1-(1-k_4)(1-e_k)^2 \\ r'_{12}=1-(1-k_4)(1-e_k) \\ r'_{13}=1-(1-k_4) \\ r'_{21}=1-(1-k_{23})(1-f_k)^2 \\ r'_{22}=1-(1-k_{22})(1-f_k) \\ r'_{23}=1-(1-k_{21}) \\ r'_3=k_{11} \\ r'_4=g_k \end{cases} \tag{3.2}$$

$$\mathscr{J}_{\mathrm{R}}^{\mathrm{K}}\begin{cases} k'_{11}=1-(1-r_{23})(1-f_r)^2 \\ k'_{12}=1-(1-r_{22})(1-f_r) \\ k'_{13}=1-(1-r_{21}) \\ k'_{21}=1-(1-r_4)(1-e_r)^2 \\ k'_{22}=1-(1-r_4)(1-e_r) \\ k'_{23}=1-(1-r_4) \\ k'_3=r_{11} \\ k'_4=g_r \end{cases}$$

where $\{e,f,g\}_k \equiv \{e,f,g\}(k_{12},k_{21},k_3)$ and $\{e,f,g\}_r \equiv \{e,f,g\}(r_{12},r_{21},r_3)$ are functions defined in Ref. 3. As in the Ising model, r_{13} and k_{13} are irrelevant. The nontrivial fixed points are at

$$(r)=(0.2955,\ 0.2306,\ 0.1596,\ 0.3237,\ 0.3019,\ 0.2794,\ 0.3911,\ 0.1806)$$

$$(k)=(0.3911,\ 0.3583,\ 0.3237,\ 0.2794,\ 0.2316,\ 0.1806,\ 0.2955,\ 0.1596)$$

for R and KD lattices, respectively. The correlation length critical exponent is found to be $\nu=1.2555$ for both lattices. The critical percolation probability for the case when bonds exist only along the edges of R tiles (i.e., $r_1=r_2$, $r_3=r_4=0$) is given by 0.4048.

4. CONCLUSION

We have extended our previous four-parameter Ising model and bond percolation model on the Penrose lattice to eight parameters and carried out the renormalization group analysis. As in any nontrivial theory, the

exact renormalization group would include an infinite number of parameters. The renormalization group based on a finite number of parameters is obtained only after some approximation. Thus, one could expect that the increase in the number of parameters would inprove the degree of approximation. The result of our analysis in generally in line with this view. For the Ising model, we obtained the specific heat critical exponent $\alpha = -0.1083$. This is a 6% improvement over our previous result, $\alpha = -0.1153$, assuming universality. In other words, the fact that we have obtained α closer to the universal value by enlarging the parameter space suggests that the exact value of α is 0, the universal value. We also note that our previous method based on the statistical average is not an over-simplification, since it gives the critical surface and the critical exponent close to the ones given by the present method.

For the bond percolation model, the present method yielded the critical exponent $\nu = 1.2555$, which should be compared with our previous value, $\nu = 1.2928$. Since the universal value is considered to be $\nu_U = 4/3$, our new value is worse than our old value. However, since the application of our method to the square lattice gave $\nu = 1.2079$, we think that the remarkable agreement of our previous data with ν_U was rather an accident. Our new value is still in agreement with ν_U within the range of error observed in the square lattice case.

Thus we conclude that our analysis strongly suggests that the Penrose lattice belongs to the universal class of regular two-dimensional lattices.

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